Poincaré gauge invariance of general relativity and Einstein-Cartan theory

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Abstract *We present a simple proof of the Poincaré gauge invariance of general relativity and Einstein-Cartan theory, in the context of the corresponding bundle of affine frames.*

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1. Introduction

It is well understood that general relativity (GR) and its extension with torsion, the Einstein-Cartan theory (E-C), are invariant under internal local Lorentz transformations ($\mathcal{L}_4$), the spin connection $\omega_{\mu ab}$ and the tetrads $e_\mu{}^a$ (coframes) (or rather the displaced fields $B_\mu{}^a = \delta_\mu{}^a - e_\mu{}^a$) being respectively the rotational and translational gravitational gauge potentials (Hehl, 1985; Hayashi, 1977). Then, the group of symmetry of both theories is the semidirect sum $\mathcal{L}_4 \odot \mathcal{D}$, with $\mathcal{D}$ the group of general coordinate transformations (O'Raifeartaigh, 1997).

However, the internal symmetry group is in fact larger, since translations $\mathcal{T}_4$ are *naturally* included, leading to $\mathcal{P}_4 = \mathcal{T}_4 \odot \mathcal{L}_4$, the Poincaré group. Then, the total symmetry of GR and E-C, as gauge theories, turns out to be $\mathcal{P}_4 \odot \mathcal{D}$ (Feynman, 1963; Hehl et al, 1976; Mc Innes, 1984; Hammond, 2002). The problem with the proof of this fact has been, historically, the apparent difficulty with the treatment of translations as part of the gauge group, that is, as *vertical* transformations of a bundle. If for a translation one writes $x^\mu \to x'^\mu = x^\mu + \xi(x)$, one is *not* considering it as a $\mathcal{P}_4$-gauge transformation, but instead as an element of $\mathcal{D}$. The appropiate treatment of gauge translations is in the framework of the *bundle of Poincaré frames* over space-time, $\mathcal{F}^P_{M^4}$: $\mathcal{P}_4 \to A^P M^4 \xrightarrow{\pi_P} M^4$.

This has been discussed by several authors (Smrz, 1977; Gronwald, 1997, 1998), and it is the purpose of this note to present an even simpler proof of this fact. On the one hand, at the global level, we show, using general theorems of connections (Kobayashi-Nomizu, 1963), that there is a 1-1 correspondence between affine Poincaré connections ω_P in $\mathcal{F}^P_{M^4}$ and pairs (θ_L, ω_L), with θ_L the canonical form and ω_L a connection on the bundle of Lorentz frames $\mathcal{F}^L_{M^4}$: $\mathcal{L}_4 \to F^L M^4 \xrightarrow{\pi_L} M^4$. On the other hand, locally, we show the invariance under $\mathcal{P}_4$-gauge transformations of the Einstein-Hilbert action for pure gravity, and the Dirac-Einstein action for the coupling of gravity to the Dirac field.

In section 2, we describe basic features of a U_4 space-time. In section 3, in the language of tetrads and spin connection, we review the E-C equations for pure gravity and for gravity coupled to the Dirac field. Lorentz and Poincaré invariance are discussed and proved in sections 4 and 5, respectively. Finally, in section 6, we discuss the nature of a shifted tetrad field, and comment on the difficulty of interpreting the theory in terms of an interaction tetrads-spin connection.

2. The space-time

We assume that space-time is a 4-dimensional Lorentzian manifold M^4 with a connection Γ compatible with the metric i.e. $D^\Gamma_\mu g_{\nu\rho} = 0$, but not necessarily symmetric: a U_4 space-time. Then, $\Gamma^\alpha_{\nu\mu} = (\Gamma_{LC})^\alpha_{\nu\mu} + K^\alpha_{\nu\mu}$, where Γ_{LC} is the Levi-Civita connection with coordinate components $(\Gamma_{LC})^\alpha_{\nu\mu} = \frac{1}{2} g^{\alpha\sigma}(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\nu\mu})$, and $K^\alpha_{\nu\mu} = (K_A)^\alpha_{\nu\mu} + (K_S)^\alpha_{\nu\mu}$ is the contortion tensor, where $(K_A)^\alpha_{\nu\mu} = T^\alpha_{\nu\mu} = -T^\alpha_{\mu\nu} = \frac{1}{2}(\Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu}) = \Gamma^\alpha_{[\mu,\nu]}$ is the *torsion* tensor, and K_S, its symmetric part, has components $(K_S)^\alpha_{\mu\nu} = g^{\alpha\rho}(T^\lambda_{\rho\mu} g_{\lambda\nu} + T^\lambda_{\rho\nu} g_{\lambda\mu})$.

In terms of the tetrads $e_a = e_a{}^\mu \partial_\mu$ and their dual coframes $e^a = e_\mu{}^a dx^\mu$, obeying $e_a{}^\mu e_\mu{}^b = \delta_a^b$ and $e_a{}^\mu e_\nu{}^a = \delta_\nu^\mu$, and the spin connection 1-form $\omega^a{}_b = \omega^a_{\mu b} dx^\mu$, Γ is given by $\Gamma^\sigma_{\mu\lambda} = e_a{}^\sigma \partial_\mu e_\lambda{}^a + e_c{}^\sigma e_\lambda{}^a \omega^c_{\mu a}$ with inverse $\omega^c_{\mu a} = e_\rho{}^c \partial_\mu e_a{}^\rho + e_\rho{}^c e_a{}^\nu \Gamma^\rho_{\mu\nu}$ (Carroll, 2004). For the metric, one has $g_{\mu\nu}(x) = \eta_{ab} e_\mu{}^a(x) e_\nu{}^b(x)$, where $x \in M^4$ and $\eta_{ab} = diag(1,-1,-1,-1)$ is the Lorentz metric. Each metric $g_{\mu\nu}$ is in 1-1 correspondence with an equivalence class of frames $[e_a{}^\mu]$: if $e_c{}'^\mu$ is in the class, then $e_a{}^\mu = h_a{}^c e_c{}'^\mu$ with $h_a{}^c$ in the Lorentz group $\mathcal{L}_4$; for the coframes $e_\mu{}^a = e_\mu{}'^c h_c^{-1\,a}$. Thus, the $e_a{}^\mu$'s and the $e_\mu{}^a$'s are both Lorentz vectors in the internal or gauge (latin) indices, and respectively vectors and 1-forms in the local coordinate (world) indices. The metric character of the connection implies $\omega_{ab} = -\omega_{ba}$ (for latin indices, $X_a = \eta_{ab} X^b$ and $X^b = \eta^{ba} X_a$). The torsion and the curvature of the connection are given by $T^a = de^a + \omega^a{}_b \wedge e^b = \frac{1}{2} T^a{}_{\mu\nu} dx^\mu \wedge dx^\nu$ with $T^a{}_{\mu\nu} = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega^a_{\mu b} e_\nu{}^b - \omega^a_{\nu b} e_\mu{}^b$, and $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu$ with $R^a_{b\mu\nu} = \partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} + \omega^a_{\mu c}\omega^c_{\nu b} - \omega^a_{\nu c}\omega^c_{\mu b}$.

3. Einstein-Cartan equations

Consider first the case of *pure gravity* ("vacuum"). The Einstein-Hilbert action is

$$S_G = \int d^4x \; eR \tag{1}$$

where $e = \sqrt{-detg_{\mu\nu}} = det(e_\nu{}^a)$, and for the Ricci scalar one has

$$R = \eta^{bc} R^a{}_{b\mu\nu} e_a{}^\mu e_c{}^\nu. \tag{2}$$

Variation of S_G with respect to the spin connection $\omega^a_{\mu b}$ and the tetrads $e_a{}^\mu$ lead, respectively, to the Cartan equation for torsion and to the Einstein equation:

$\delta_\omega S_G = 0 \Longrightarrow$

$$T^\nu_{ac} + e_a{}^\nu T_c - e_c{}^\nu T_a = 0 \tag{3}$$

or

$$T^\nu_{\rho\sigma} + \delta^\nu_\rho T_\sigma - \delta^\nu_\sigma T_\rho = 0. \tag{3a}$$

$\delta_e S_G = 0 \Longrightarrow$

$$G^a{}_\mu = 0 \tag{4}$$

with

$$G^a{}_\mu = R^a{}_\mu - \frac{1}{2} R e_\mu{}^a, \tag{5}$$

where $R^a{}_\mu = \eta^{ab} R_{b\mu} = \eta^{ab} R^c{}_{b\nu\mu} e_c{}^\nu$. In vacuum $R = 0$, then

$$R^a{}_\mu = 0. \tag{5a}$$

In this case, *torsion vanishes*, since taking the $\nu - \sigma$ trace in (3a), for the torsion vector one obtains $T_\rho = T^\nu_{\rho\nu} = 0$ and therefore, by (3a) again,

$$T^\mu_{\nu\rho} = 0. \tag{6}$$

Thus, for the pure gravity case, E-C theory reduces to GR.

The coupling of gravity to Dirac fermions is described by the action

$$S_{D-E} = k \int d^4x \; eL_{D-E} = k \int d^4x \; e \; (\frac{i}{2}(\bar\psi \gamma^a (D_a \psi) - (\bar D_a \bar\psi) \gamma^a \psi) - m\bar\psi\psi) \tag{7}$$

where

$$D_a \psi = (e_a - \frac{i}{4}\omega_{abc}\sigma^{bc})\psi = e_a{}^\mu(\partial_\mu - \frac{i}{4}\omega_{\mu bc}\sigma^{bc})\psi = e_a{}^\mu D_\mu \psi \tag{8}$$

and

$$\bar{D}_a\bar{\psi} = e_a\bar{\psi} + \frac{i}{4}\omega_{abc}\bar{\psi}\sigma^{bc} = e_a{}^{\mu}(\partial_\mu\bar{\psi} + \frac{i}{4}\omega_{\mu bc}\bar{\psi}\sigma^{bc}) = e_a{}^{\mu}\bar{D}_\mu\bar{\psi} \tag{9}$$

are the covariant derivatives of the Dirac field ψ and its conjugate $\bar{\psi} = \psi^\dagger\gamma_0$ with respect to the spin connection, which give the *minimal coupling* between fermions and gravity. $\sigma^{bc} = \frac{i}{2}[\gamma^b, \gamma^c]$, and the γ^a's are the usual numerical (constant) Dirac gamma matrices satisfying $\{\gamma^a, \gamma^b\} = 2\eta^{ab}I$, $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{j\dagger} = -\gamma^j$. $k = -16\pi\frac{G}{c^4}$. Variation with respect to the spin connection,

$$\delta_\omega S_{D-E} = \frac{k}{8}\int d^4x\ e\ \bar{\psi}\{\gamma^\mu, \sigma^{bc}\}\psi\delta\omega_{\mu bc} = \frac{k}{2}\int d^4x\ e\ S^{\mu bc}\delta\omega_{\mu bc}$$

with $S^{\mu bc} = e_a{}^{\mu}S^{abc}$, where

$$S^{abc} = \frac{1}{4}\bar{\psi}\{\gamma^a, \sigma^{bc}\}\psi \tag{10}$$

is the *spin density tensor* of the Dirac field. S^{abc} is totally antisymmetric and therefore has 4 independent components: S^{012}, S^{123}, S^{230} and S^{301}.

Combining this result with the corresponding variation for the pure gravitational field, we obtain

$$0 = \delta_\omega(S_G + S_{D-E}) = \int d^4x\ e\ \delta\omega_\nu{}^{ac}(T^\nu_{ac} + e_a{}^{\nu}T_c - e_c{}^{\nu}T_a + \frac{k}{2}S^\nu_{ac}) \tag{11}$$

and therefore

$$T^\nu_{ac} + e_a{}^{\nu}T_c - e_c{}^{\nu}T_a = -\frac{k}{2}S^\nu_{ac},$$

the *Cartan equation*. Multiplying by $e_\rho{}^a e_\sigma{}^c$ one obtains

$$T^\nu_{\rho\sigma} + \delta^\nu_\rho T_\sigma - \delta^\nu_\sigma T_\rho = -\frac{k}{2}S^\nu_{\rho\sigma} \tag{12}$$

with

$$S^\nu_{\rho\sigma} = \frac{1}{4}\bar{\psi}\{\gamma^\mu, \sigma_{\rho\sigma}\}\psi.$$

The solution of (12) gives the *torsion in terms of the spin tensor*:

$$T^\nu_{\rho\sigma} = \frac{8\pi G}{c^4}(S^\nu_{\rho\sigma} + \frac{1}{2}(\delta^\nu_\rho S_\sigma - \delta^\nu_\sigma S_\rho)) \tag{13}$$

with $S_\rho = S^\nu_{\rho\nu}$. (In natural units, $G = c = \hbar = 1$ and so $T^\nu_{\rho\sigma} = 8\pi(S^\nu_{\rho\sigma} + \frac{1}{2}(\delta^\nu_\rho S_\sigma - \delta^\nu_\sigma S_\rho))$.)

Finally, variation with respect to the tetrads,

$$\delta_e S_{D-E} = k\int d^4x\ e\ (\frac{i}{2}(\bar{\psi}\gamma^a(D_\mu\psi) - (\bar{D}_\mu\bar{\psi})\gamma^a\psi) - e_\mu{}^a L_{D-E})\delta e_a{}^{\mu}.$$

For the Dirac fields which obey the equations of motion

$$\frac{\delta S_{D-E}}{\delta\bar{\psi}} = \frac{\delta S_{D-E}}{\delta\psi} = 0$$

i.e.

$$i\gamma^a(\bar{D}_a\bar{\psi}) + m\bar{\psi} = i\gamma^a D_a\psi - m\psi = 0$$

the Dirac-Einstein lagrangian vanishes i.e. $L_{D-E}|_{eq.\ mot.} = 0$. Then, combining this result with the corresponding variation for the pure gravitational field,

$$0 = \delta_e(S_G + S_{D-E}) = \int d^4x\ e\ (2R^a{}_\mu - Re_\mu{}^a + k\frac{i}{2}(\bar{\psi}\gamma^a(D_\mu\psi) - (\bar{D}_\mu\bar{\psi})\gamma^a\psi))\delta e_a{}^\mu, \tag{14}$$

and from the arbitrariness of $\delta e_a{}^\mu$,

$$R^a{}_\mu - \frac{1}{2}Re_\mu{}^a = -\frac{k}{2}T^a{}_\mu \tag{15}$$

with

$$T^a{}_\mu = \frac{i}{2}(\bar{\psi}\gamma^a(D_\mu\psi) - (\bar{D}_\mu\bar{\psi})\gamma^a\psi) \tag{16}$$

the *energy-momentum tensor* of the Dirac field. Multiplying (15) by $e_a{}^\nu$ one obtains

$$R^\nu{}_\mu - \frac{1}{2}R\delta^\nu_\mu = -\frac{k}{2}T^\nu{}_\mu \quad or \quad R_{\lambda\mu} - \frac{1}{2}Rg_{\lambda\mu} = -\frac{k}{2}T_{\lambda\mu}, \tag{15a}$$

the *Einstein equation* in local coordinates.

Note: For L_{D-E} one has

$$L_{D-E} = e_a{}^\mu T^a{}_\mu - m\bar{\psi}\psi$$

i.e. $T^a{}_\mu$ *couples to the tetrad*. On the other hand,

$$T^a{}_\mu = \theta^a{}_\mu + \omega_{\mu bc}S^{abc}$$

where

$$\theta^a{}_\mu = \frac{i}{2}(\bar{\psi}\gamma^a\partial_\mu\psi - (\partial_\mu\bar{\psi})\gamma^a\psi)$$

is the *canonical energy-momentum tensor* of the Dirac field. Then,

$$L_{D-E} = e_a{}^\mu\theta^a{}_\mu + e_a{}^\mu\omega_{\mu bc}S^{abc} - m\bar{\psi}\psi = e_a{}^\mu\theta^a{}_\mu + \omega_{abc}S^{abc} - m\bar{\psi}\psi.$$

So, $\theta^a{}_\mu$ couples to the tetrad while spin couples to the spin connection; moreover, since S^{abc} is totally antisymmetric, *the Dirac field only interacts with the totally antisymmetric part of the connection.*

4. Lorentz gauge invariance

Under local Lorentz transformations $h_a{}^b(x)$, tetrads and coframes transform as indicated in section 2; as a consequence, the coordinate invariant volume element $d^4x\ e$ is also gauge invariant: in fact,

$$g_{\mu\nu}(x) = \eta_{ab}e_\mu{}^a(x)e_\nu{}^b(x) = \eta_{ab}e'^{\ c}_\mu h^{-1}_c{}^a e'^{\ d}_\nu h^{-1}_d{}^b = e'^{\ c}_\mu e'^{\ d}_\nu h^{-1}_c{}^a\eta_{ab}h^{-1}_d{}^b = e'^{\ c}_\mu e'^{\ d}_\nu\eta_{cd} = g'_{\mu\nu}(x)$$

implies $e'(x) = e(x)$, and since $x'^\mu = x^\mu$, then $d^4x\ e = d^4x'\ e'$.

On the other hand, the transformation of the spin connection is given by

$$\omega^c{}_a = h_c{}^d\omega'^r{}_d h^{-1}_r{}^c + (dh_a{}^d)h^{-1}_d{}^c, \tag{16}$$

which is not a Lorentz tensor. Its curvature, however, is a Lorentz tensor:

$$R^a{}_b = h_b{}^d h^{-1}_c{}^a R'^c{}_d, \tag{17}$$

and therefore the Ricci scalar is also gauge invariant:

$$R = R^a{}_b e_a\eta^{bc}e_c = h_b{}^d R'^c{}_d h^{-1}_c{}^a h_a{}^f e'_f\eta^{bg}h_g{}^l e'_l = R'^c{}_d\delta^f_c e'_f\eta^{dl}e'_l = R'^c{}_d e'_c\eta^{dl}e'_l = R'. \tag{18}$$

Then, S_G is Lorentz gauge invariant. (A direct and more explicit proof of the gauge invariance of R is given in Appendix 1.)

The part of the action corresponding to the coupling of gravity to the Dirac field, S_{D-E}, is automatically local Lorentz invariant, since it is written in terms of the covariant derivatives $D_a\psi$ and $\bar{D}_a\bar{\psi}$.

5. Poincaré gauge invariance

5.1. *Global analysis*

The affine group $GA_4(\mathbb{R}) = \{\begin{pmatrix} g & \xi \\ 0 & 1 \end{pmatrix},\ g \in GL_4(\mathbb{R}),\ \xi \in \mathbb{R}^4\}$ acts on the affine space $\mathbb{A}^4 = \{\begin{pmatrix} \lambda \\ 1 \end{pmatrix},\ \lambda \in \mathbb{R}^4\}$ in the form

$$GA_4(\mathbb{R}) \times \mathbb{A}^4 \to \mathbb{A}^4,\ (\begin{pmatrix} g & \xi \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda \\ 1 \end{pmatrix}) \mapsto \begin{pmatrix} g\lambda + \xi \\ 1 \end{pmatrix}. \tag{19}$$

Then, one has the following diagram of short exact sequences (s.e.s.'s) of groups and group homomorphisms:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R}^4 & \stackrel{\mu}{\longrightarrow} & GA_4(\mathbb{R}) & \underset{\stackrel{\longleftarrow}{\rho}}{\stackrel{\nu}{\longrightarrow}} & GL_4(\mathbb{R}) & \longrightarrow & 0 \\
 & & Id\uparrow & & \uparrow\iota & & \uparrow\iota & & \\
0 & \longrightarrow & \mathbb{R}^4 & \stackrel{\mu|}{\longrightarrow} & \mathcal{P}_4 & \underset{\stackrel{\longleftarrow}{\rho|}}{\stackrel{\nu|}{\longrightarrow}} & \mathcal{L}_4 & \longrightarrow & 0
\end{array}$$

with $\mu(\xi) = \begin{pmatrix} I_4 & \xi \\ 0 & 1 \end{pmatrix}$ and $\nu(\begin{pmatrix} g & \lambda \\ 0 & 1 \end{pmatrix}) = g$. μ is 1-1, ν is onto, and $ker(\nu) = Im(\mu) = \{\begin{pmatrix} I_4 & \xi \\ 0 & 1 \end{pmatrix},\ \xi \in \mathbb{R}^4\}$. We have also restricted μ and ν (respectively $\mu|$ and $\nu|$) to the connected components of the Poincaré ($\mathcal{P}_4$) and Lorentz ($\mathcal{L}_4$) groups. Both s.e.s.'s split, i.e. there exists the group homomorphism $\rho : GL_4(\mathbb{R}) \to GA_4(\mathbb{R})$, $g \mapsto \rho(g) = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ and its restriction $\rho|$ to $\mathcal{L}_4$, such that $\nu \circ \rho = Id_{GL_4(\mathbb{R})}$ and $\nu| \circ \rho| = Id_{\mathcal{L}_4}$. So

$$GA_4(\mathbb{R}) = \mathbb{R}^4 \odot GL_4(\mathbb{R}),\ \mathcal{P}_4 = \mathbb{R}^4 \odot \mathcal{L}_4 \tag{20}$$

with composition law

$$(\lambda', g')(\lambda, g) = (\lambda' + g'\lambda, g'g). \tag{20a}$$

The above s.e.s.'s pass to s.e.s.'s of the corresponding Lie algebras:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R}^4 & \stackrel{\tilde{\mu}}{\longrightarrow} & ga_4(\mathbb{R}) & \underset{\stackrel{\longleftarrow}{\tilde{\rho}}}{\stackrel{\tilde{\nu}}{\longrightarrow}} & gl_4(\mathbb{R}) & \longrightarrow & 0 \\
 & & Id\uparrow & & \uparrow\iota & & \uparrow\iota & & \\
0 & \longrightarrow & \mathbb{R}^4 & \stackrel{\tilde{\mu}|}{\longrightarrow} & p_4 & \underset{\stackrel{\longleftarrow}{\tilde{\rho}|}}{\stackrel{\tilde{\nu}|}{\longrightarrow}} & l_4 & \longrightarrow & 0
\end{array}$$

with $gl_4(\mathbb{R}) = \mathbb{R}(4)$, $ga_4(\mathbb{R}) = \mathbb{R}^4 \odot gl_4(\mathbb{R})$ with Lie product

$$(\lambda', R')(\lambda, R) = (R'\lambda - R\lambda', [R', R]), \tag{21}$$

where $[R', R]$ is the Lie product in $gl_4(\mathbb{R})$ and $[\lambda', \lambda] = 0$ in $\mathbb{R}^4$, $\tilde{\mu}(\xi) = (\xi, 0)$, $\tilde{\nu}(\xi, R) = R$, and $\tilde{\rho}(R) = (0, R)$. $\tilde{\mu}$, $\tilde{\nu}$ and $\tilde{\rho}$ (and their corresponding restrictions $\tilde{\mu}|$, $\tilde{\nu}|$ and $\tilde{\rho}|$) are Lie algebra homomorphisms, with $\tilde{\nu} \circ \tilde{\rho} = Id_{gl_4(\mathbb{R})}$ and $\tilde{\nu}| \circ \tilde{\rho}| = Id_{l_4}$. The s.e.s.'s split only at the level of vector spaces i.e. if $(\lambda, R) \in ga_4(\mathbb{R})$, then $(\lambda, R) = \tilde{\mu}(\lambda) + \tilde{\rho}(R)$, but $(\lambda, R) \neq \tilde{\mu}(\lambda)\tilde{\rho}(R)$.

If $\mathcal{F}_{M^4} : GL_4 \to FM^4 \xrightarrow{\pi_F} M^4$ and $\mathcal{A}_{M^4} : GA_4 \to AM^4 \xrightarrow{\pi_A} M^4$ are respectively the bundles of linear and affine frames over M^4, where $FM^4 = \cup_{x \in M^4}(\{x\} \times (FM^4)_x)$ with $(FM^4)_x$ the set of ordered basis

$r_x = (v_{1x}, \ldots, v_{4x})$ of T_xM^4, and $AM^4 = \cup_{x\in M^4}(\{x\} \times AM^4_x)$ with $AM^4_x = \{(v_x, r_x),\ v_x \in A_xM^4,\ r_x \in (FM^4)_x\}$, where A_xM^4 is the tangent space at x considered as an affine space (Appendix 3), then one has the following bundle homomorphism:

$$\begin{array}{ccc}
AM^4 \times GA_4 & \overset{\beta\times\nu}{\underset{\gamma\times\rho}{\rightleftarrows}} & FM^4 \times GL_4 \\
\psi_A \downarrow & & \downarrow \psi_F \\
AM^4 & \overset{\beta}{\underset{\gamma}{\rightleftarrows}} & FM^4 \\
\pi_A \downarrow & & \downarrow \pi_F \\
M^4 & \xrightarrow{Id} & M^4
\end{array}$$

where $\beta(x, (v_x, r_x)) = (x, r_x),\ \gamma(x, r_x) = (x, (0_x, r_x)),\ 0 \in T_xM^n,\ \psi_F((x, r_x), g) = (x, r_xg)$, and

$$\psi_A((x, (v_x, r_x)), (\xi, g)) = (x, (v_x + r_x\xi, r_xg)). \tag{22}$$

A *general affine connection* (g.a.c.) on M^4 is a connection in the bundle of affine frames $\mathcal{A}_{M^4}$; let ω_A be the 1-form of this connection, then $\omega_A \in \Gamma(T^*AM^4 \otimes ga_4)$. From the smoothness of γ, the pull-back $\gamma^*(\omega_A)$ is a ga_4-valued 1-form on FM^n:

$$\gamma^*(\omega_A) = \varphi \odot \omega_F, \tag{23}$$

where ω_F is a connection on FM^4, and φ is an $\mathbb{R}^4$-valued 1-form. There is a 1-1 correspondence between g.a.c.'s on AM^4 and pairs (ω_F, φ) on FM^4:

$$\{\omega_A\}_{g.a.c.} \longleftrightarrow \{(\omega_F, \varphi)\}. \tag{24}$$

ω_A is an *affine connection* (a.c.) on M^4 if φ is the soldering (canonical) form θ_{FM^4} (see Appendix 2) on FM^4. Then, if ω_A is an a.c. on AM^4,

$$\gamma^*(\omega_A) = \theta_{FM^4} \odot \omega_F. \tag{25}$$

There is then a 1-1 correspondence

$$\{\omega_A\}_{a.c.} \longleftrightarrow \{\omega_F\}, \tag{26}$$

since θ_{FM^4} is fixed. Also, if Ω_A is the curvature of ω_A, then

$$\gamma^*(\Omega_A) = D^{\omega_F}\theta \odot \Omega_F = T_F \odot \Omega_F \tag{27}$$

since $D^{\omega_F}\theta_{FM^4} = T_F$: the *torsion* of the connection ω_F on FM^4.

We now consider the following diagram of bundle homomorphisms:

$$\begin{array}{ccccccc}
AM^4 \times GA_4 & \xleftarrow{\iota\times\iota} & A^PM^4 \times \mathcal{P}_4 & \overset{\beta|\times\nu|}{\underset{\gamma|\times\rho|}{\rightleftarrows}} & F^LM^4 \times \mathcal{L}_4 & \xrightarrow{\iota\times\iota} & FM^4 \times GL_4 \\
\psi_A \downarrow & & \psi_A| \downarrow & & \downarrow \psi_F| & & \downarrow \psi_F \\
AM^4 & \xleftarrow{\iota} & A^PM^4 & \overset{\beta|}{\underset{\gamma|}{\rightleftarrows}} & F^LM^4 & \xrightarrow{\iota} & FM^4 \\
\pi_A \downarrow & & \pi_A| \downarrow & & \downarrow \pi_F| & & \downarrow \pi_F \\
M^4 & \xrightarrow{Id} & M^4 & \xrightarrow{Id} & M^4 & \xrightarrow{Id} & M^4
\end{array}$$

where $\pi_A| = \pi_P$, $\pi_F| = \pi_L$, $\psi_A| = \psi_P$ and $\psi_F| = \psi_L$, where ψ_P and ψ_L are the group actions in the Poincaré and Lorentz bundles, respectively.

The facts that $A^P M^4$ is a subbundle of AM^4 and $F^L M^4$ is a subbundle of FM^4, with structure groups and Lie algebras the corresponding subgroups and sub-Lie algebras, and the existence of the restrictions $\beta| : A^P M^4 \to F^L M^4$ and $\gamma| : F^L M^4 \to A^P M^4$, allow us to obtain similar conclusions for the relations between affine connections on the Poincaré bundle and linear connections on the Lorentz bundle:

There is a 1-1 correspondence between affine Poincaré connections ω_P on $F^P M^4$ and Lorentz connections on $F^L M^4$:

$$\{\omega_P\} \longleftrightarrow \{\omega_L\} \tag{28}$$

with

$$\gamma|^*(\omega_P) = \theta_L \odot \omega_L \tag{29}$$

where $\theta_L = \theta_{FM^4}|_{F^L M^4}$ is the canonical form on $F^L M^4$. Also,

$$\gamma|^*(\Omega_P) = D^{\omega_L}\theta_L \odot \Omega_L = T_L \odot \Omega_L. \tag{30}$$

So, there is a 1-1 correspondence between curvatures of affine connections on $F^P M^4$ and torsion and curvature pairs on $F^L M^4$:

$$\{\Omega_P\} \longleftrightarrow \{(T_L, \Omega_L)\}. \tag{31}$$

For pure gravity governed by the Einstein-Hilbert action, $T_L = 0$.

5.2. *Local analysis: invariance of the actions S_G and S_{D-E}*

To explicitly prove the Poincaré gauge invariance of GR and E-C theory, we have to consider as gauge transformations both the Lorentz part, already studied in the previous section, and the translational part. This last has to be done using the bundle of Poincaré frames $\mathcal{F}^P_{M^4}$.

A gauge transformation or vertical automorphism in an arbitrary principal G-bundle $\xi : G \to P \xrightarrow{\pi} B$, is a diffeomorphism $\alpha : P \to P$ such that i) $\alpha(pg) = \alpha(p)g$ and ii) $\pi(\alpha(g)) = \pi(p)$, for all $p \in P$ and $g \in G$. Therefore, from ii), $\alpha(p) = pk$ for some $k \in G$. Then there is a bijection $Aut_{vert}(P) \xrightarrow{\Phi} \Gamma_{eq}(P, G)$, $\Phi(\alpha) = \gamma_\alpha$ with $\alpha(p) = p\gamma_\alpha(p)$ and $\gamma_\alpha(pg) = g^{-1}\gamma_\alpha(p)g$; for the inverse, $\gamma \mapsto \alpha_\gamma$ with $\alpha_\gamma(p) = p\gamma(p)$.

The action of $\mathcal{P}_4$ on $A^P M^4$ is given by

$$\psi_P : A^P M^4 \times \mathcal{P}_4 \to A^P M^4,\ \psi_P((x,(v_x, r_x)),(\xi, h)) \equiv (x,(v_x, r_x))(\xi, h) = (x,(v_x + r_x\xi, r_x h))$$

$$= (x,(v'_x, r'_x)), \tag{32}$$

where $r_x = (e_{ax})$, $a = 1,2,3,4$, is a Lorentz frame, $h \in \mathcal{L}_4$, and $\xi \in \mathbb{R}^4 \cong \mathbb{R}^{1,3}$ is a Poincaré gauge translation. For a pure translation, $h = I_L$ i.e. ${h_a}^b = {\delta_a}^b$ and therefore

$$(x,(v_x, r_x))(\xi, I_L) = (x,(v_x + r_x\xi, r_x I_L)) = (x,(v_x + r_x\xi, r_x))$$

i.e.

$$r'_x = r_x. \tag{33}$$

Therefore $e'_{ax} = e_{ax}$, $a = 1,2,3,4$, and then, from the definition of $\omega^a_{\mu b}$ in section 2,

$$\omega'^a_{\mu b} = \omega^a_{\mu b} \tag{34}$$

since $\Gamma^\mu_{\nu\rho} = (\Gamma_{LC})^\mu_{\nu\rho} + K^\mu_{\nu\rho}$ remains unchanged (in the case of pure gravity $K^\mu_{\nu\rho} = 0$). So the coordinate Ricci scalar R is also a gauge scalar, and therefore S_G is invariant.

By the same reason invoked in the case of Lorentz gauge invariance, S_{D-E} is also invariant under translations: in an arbitrary G-bundle P with connection ω, a section s of an associated bundle and its covariant derivative $D^\omega s$ transform in the same way.

The Poincaré bundle extends the symmetry group of GR and E-C theory to the semidirect sum

$$G_{GR/E-C} = \mathcal{P}_4 \odot \mathcal{D}, \tag{35}$$

with composition law

$$((\xi', h'), g')((\xi, h), g) = ((\xi', h')(g'(\xi, h)g'^{-1}), g'g). \tag{35a}$$

The left action of $\mathcal{D}$ on $\mathcal{P}_4$ is given by the commutative diagram

$$\begin{array}{ccc} A^P M^4 & \stackrel{(\xi,h)}{\longrightarrow} & A^P M^4 \\ g\downarrow & & \downarrow g \\ A^P M^4 & \stackrel{(\xi',h')}{\longrightarrow} & A^P M^4 \end{array}$$

with

$$g: A^P M^4 \to A^P M^4,\ (x, (v_x^\mu \frac{\partial}{\partial x^\mu}|_x, (e_{ax}{}^\nu \frac{\partial}{\partial x^\nu}|_x))) \mapsto (x, (v_x'^\mu \frac{\partial}{\partial x'^\mu}|_x, (e_{ax}{}'^\nu \frac{\partial}{\partial x'^\nu}))), \tag{36}$$

where $v_x'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha}|_x v_x^\alpha$ and $e_{ax}{}'^\nu = \frac{\partial x'^\nu}{\partial x^\beta}|_x e_{ax}{}^\beta$.

6. Gravitational potentials and interactions

It is usually said that the coframes $e^a = e_\mu{}^a dx^\mu$ are the translational gravitational potentials (Hehl, 1985; Hehl et al, 1976; Hammond, 2002). This is *not* strictly true since these fields are not gauge potentials, but tensors, both in their Lorentz (a) and world (μ) indices: see section 2 and (Hayashi, 1977). The translational gauge potentials are the 1-form fields $B_\mu{}^a$ locally defined as follows (Hayashi and Nakano, 1967; Aldrovandi and Pereira, 2007):

$$B_\mu{}^a = e_\mu{}^a - \frac{\partial v_x^a}{\partial x^\mu} \quad or \quad B^a = e^a - dv_x^a, \tag{37}$$

where $v_x = \sum_{a=0}^3 v_x^a e_{ax} \in A_x M^4$ (section 5.1.); the v_x^a's are here considered the coordinates of the tangent space at x. A straightforward calculation leads to the following transformation properties:

Internal Lorentz:

$$B_\mu{}'^a = h_a{}^b B_\mu{}^b - \partial_\mu(h_b{}^a) v_x^b \quad or \quad B'^a = h_b{}^a B^b - (dh_b{}^a) v_x^b, \tag{38}$$

General coordinate transformations:

$$B_\mu{}'^a = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu{}^a, \tag{39}$$

Internal translations:

$$B_\mu{}'^a = B_\mu{}^a - \partial_\mu \xi^a \quad or \quad B'^a = B^a - d\xi^a. \tag{40}$$

Then, $B = B_\mu dx^\mu = B_\mu{}^a dx^\mu b_a$, where b_a, $a = 0, 1, 2, 3$, is the canonical basis of $\mathbb{R}^4$, is the connection 1-form corresponding to the translations.

In terms of the $B_\mu{}^a$ fields and the spin connection, the Ricci scalar (2) is given by

$$R = (\frac{\partial v_x^a}{\partial x^\mu}\frac{\partial v_x^b}{\partial x^\nu} + \frac{\partial v_x^a}{\partial x^\mu} B_\nu{}^b + \frac{\partial v_x^b}{\partial x^\nu} B_\mu{}^a + B_\mu{}^a B_\nu{}^b)(\partial^\mu \omega_{ab}^\nu - \partial^\nu \omega_{ab}^\mu + \omega_{ac}^\mu \omega_b^{\nu c} - \omega_{ac}^\nu \omega_b^{\mu c}). \tag{41}$$

If one intends to use this Lagrangian density as describing a $({B_\mu}^a, \omega^\nu_{bc})$ (or $({e_\mu}^a, \omega^\nu_{bc})$) interaction (Randono, 2010), then immediately faces the problem that the ${B_\mu}^a$ (or ${e_\mu}^a$) does not have a free part (in particular a kinematical part), since all its powers are multiplied by ω's or $\partial\omega$'s. So an interpretation in terms of fields interaction seems difficult, and may be, impossible.

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Appendix 1

The Ricci scalar is given by

$$R = \eta^{bd} e_a{}^{\mu} e_d{}^{\nu} (\partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} + \omega^a_{\mu c}\omega^c_{\nu b} - \omega^a_{\nu c}\omega^c_{\mu b}) \equiv \eta^{bd} e_a{}^{\mu} e_d{}^{\nu} ((\gamma) - (\delta) + (\alpha) - (\beta)),$$

with $(\gamma) = \partial_\mu \omega^a_{\nu b}$, $(\delta) = \partial_\nu \omega^a_{\mu b}$, $(\alpha) = \omega^a_{\mu c}\omega^c_{\nu b}$, and $(\beta) = \omega^a_{\nu c}\omega^c_{\mu b}$.

Under the transformation

$$\omega^a_{\mu c} = h_c{}^l \omega'^r_{\mu l} h_r^{-1\,a} + (\partial h_c{}^l) h_l^{-1\,a}$$

we have:

$(\alpha) = (a) + (b) + (c) + (d)$ with

$$(a) = h_c{}^l \omega'^r_{\mu l} h_r^{-1\,a} h_b{}^g \omega'^s_{\nu g} h_s^{-1\,c},\ (b) = h_c{}^l \omega'^r_{\mu l} h_r^{-1\,a} (\partial_\nu h_b{}^g) h_g^{-1\,c},$$

$$(c) = h_b{}^g \omega'^s_{\nu g} h_s^{-1\,c} (\partial_\mu h_c{}^l) h_l^{-1\,a},\ (d) = (\partial_\mu h_c{}^l) h_l^{-1\,a} (\partial_\nu h_b{}^g) h_g^{-1\,c};$$

$(\beta) = (e) + (f) + (g) + (h)$ with

$$(e) = h_c{}^g \omega'^s_{\nu g} h_s^{-1\,a} h_b{}^l \omega'^r_{\mu l} h_r^{-1\,c},\ (f) = h_c{}^g \omega'^s_{\nu g} h_s^{-1\,a} (\partial_\mu h_b{}^l) h_l^{-1\,c},$$

$$(g) = h_b{}^l \omega'^r_{\mu l} h_r^{-1\,c} (\partial_\nu h_c{}^g) h_g^{-1\,a},\ (h) = (\partial_\nu h_c{}^l) h_l^{-1\,a} (\partial_\mu h_b{}^g) h_g^{-1\,c};$$

$(\gamma) = [1] + [2] + [3] + [4]$ with

$$[1] = h_b{}^n h_t^{-1\,a} (\partial_\mu \omega'^t_{\nu n}),\ [2] = \omega'^t_{\nu n} \partial_\mu (h_b{}^n h_t^{-1\,a}),\ [3] = (\partial_\mu \partial_\nu h_b{}^n) h_n^{-1\,a},\ [4] = (\partial_\nu h_b{}^n)(\partial_\mu h_n^{-1\,a});$$

and $(\delta) = [5] + [6] + [7] + [8]$ with

$$[5] = h_b{}^l h_s^{-1\,a} (\partial_\nu \omega'^s_{\mu l}),\ [6] = \omega'^s_{\mu l} \partial_\nu (h_b{}^l h_s^{-1\,a}),\ [7] = (\partial_\nu \partial_\mu h_b{}^l) h_l^{-1\,a},\ [8] = (\partial_\mu h_b{}^l)(\partial_\nu h_l^{-1\,a}).$$

Now,

$[3] - [7] = (\partial_\mu \partial_\nu h_b{}^n) h_n^{-1\,a} - (\partial_\nu \partial_\mu h_b{}^l) h_l^{-1\,a} = 0,$

$(b) + (c) = \omega'^r_{\mu l} h_r^{-1\,a} \partial_\nu h_b{}^l - \omega'^s_{\nu g} h_b{}^g \partial_\mu h_s^{-1\,a},$

$(f) + (g) = \omega'^s_{\nu g} h_s^{-1\,a} \partial_\mu h_b{}^g - \omega'^r_{\mu l} h_b{}^l \partial_\nu h_r^{-1\,a};$

so

$((b) + (c)) - ((f) + (g)) = \omega'^r_{\mu l} \partial_\nu (h_r^{-1\,a} h_b{}^l) - \omega'^s_{\nu g} \partial_\mu (h_s^{-1\,a} h_b{}^g);$

also,

$[2] - [6] = \omega'^s_{\nu g} \partial_\mu (h_b{}^g h_s^{-1\,a}) - \omega'^r_{\mu l} \partial_\nu (h_b{}^l h_r^{-1\,a});$

then

$((b) + (c)) - ((f) + (g)) + ([2] - [6]) = 0.$

Also,

$[4]-[8]=(\partial_\nu h_b{}^l)(\partial_\mu h^{-1}_l{}^a)-(\partial_\mu h_b{}^l)(\partial_\nu h^{-1}_l{}^a)$

and

$(d)-(h)=(\partial_\nu h^{-1}_l{}^a)(\partial_\mu h_b{}^l)-(\partial_\mu h^{-1}_l{}^a)(\partial_\nu h_b{}^l);$

so

$([4]-[8])+((d)-(h))=0.$

Finally,

$[1]-[5]+(a)-(e)=h_b{}^l h^{-1}_s{}^a(\partial_\mu\omega'^s_{\nu l}-\partial_\nu\omega'^s_{\mu l}+\omega'^s_{\mu r}\omega'^r_{\nu l}-\omega'^s_{\nu r}\omega'^r_{\mu l}).$

Therefore,

$R=\eta^{bd}e_a{}^\mu e_d{}^\nu h_b{}^l h^{-1}_s{}^a(\partial_\mu\omega'^s_{\nu l}-\partial_\nu\omega'^s_{\mu l}+\omega'^s_{\mu r}\omega'^r_{\nu l}-\omega'^s_{\nu r}\omega'^r_{\mu l})=\eta^{lt}e_s{}'^\mu e_t{}'^\nu(\partial_\mu\omega'^s_{\nu l}-\partial_\nu\omega'^s_{\mu l}+\omega'^s_{\mu r}\omega'^r_{\nu l}-\omega'^s_{\nu r}\omega'^r_{\mu l})$

$=R'.$

Appendix 2

The *soldering* or *canonical* form on the frame bundle $\mathcal{F}_{M^n}$ of an n dimensional differentiable manifold, is the $\mathbb{R}^n$-valued differential 1-form on FM^n given by

$$\theta:FM^n\to T^*FM^n\otimes\mathbb{R}^n,\ (x,r_x)\mapsto\theta((x,r_x))=((x,r_x),\theta_{(x,r_x)}),$$

with

$$\theta_{(x,r_x)}:T_{(x,r_x)}FM^n\to\mathbb{R}^n,\ v_{(x,r_x)}\mapsto\theta_{(x,r_x)}(v_{(x,r_x)})=\tilde{r}_x^{-1}\circ d\pi_F|_{(x,r_x)}(v_{(x,r_x)})$$

i.e.

$$\theta_{(x,r_x)}=\tilde{r}_x^{-1}\circ d\pi_F|_{(x,r_x)},$$

where π_F is the projection in the bundle $\mathcal{F}_{M^n}:GL_n(\mathbb{R})\to FM^n\xrightarrow{\pi_F}M^n$ and $\tilde{r}_x$ is the vector space isomorphism

$$\tilde{r}_x:\mathbb{R}^n\to T_xM,\ (\lambda^1,\ldots,\lambda^n)\mapsto\tilde{r}_x(\lambda^1,\ldots,\lambda^n)=\sum_{i=1}^n\lambda^i v_{ix}$$

with inverse

$$\tilde{r}_x^{-1}(\sum_{i=1}^n\lambda^i v_{ix})=(\lambda^1,\ldots,\lambda^n).$$

In local coordinates (x^ρ,X^μ_ν) on $\mathcal{F}_U$,

$$\theta^\mu=\sum_{\nu=1}^n(X^{-1})^\mu_\nu dx^\nu$$

with $(X^{-1})^\mu_\nu(x,r_x)=(X^\mu_\nu(x,r_x))^{-1}=(v^\mu_{\nu x})^{-1}$, where $r_x=(v_{1x},\ldots,v_{nx})$ and $v_{\nu x}=\sum_{\mu=1}^n v^\mu_{\nu x}\frac{\partial}{\partial x^\mu}|_x$. Then $\theta^a=e_\mu{}^a\theta^\mu=e_\mu{}^a(X^{-1})^\mu{}_\nu dx^\nu=(X^{-1})^a{}_\nu dx^\nu=e_\nu{}^a dx^\nu=e^a$; so, if ω_F is a connection on $\mathcal{F}_{M^n}$, then $D^{\omega_F}\theta^a=d\theta^a+\omega^a_{F\,b}e^b=T^a_F$ is the torsion of ω_F.

Appendix 3

An *affine space* is a triple (V, φ, A) where V is a vector space, A is a set, and φ is a free and transitive left action of V as an additive group on A:

$$\varphi : V \times A \to A,\ (v, a) \mapsto v + a,$$

with

$$0 + a = a \textit{ and } (v_1 + v_2) + a = v_1 + (v_2 + a),\ \textit{for all } a \in A \textit{ and all } v_1, v_2 \in V.$$

Then, given $a, a' \in A$, there exists a unique $v \in V$ such that $a' = v + a$. Also, if v_0 is fixed in V, $\varphi_{v_0} : A \to A$, $\varphi_{v_0}(a) = \varphi(v_0, a)$ is a bijection.

Example. $A = V$: The vector space itself is considered as the set on which V acts. In particular, when $V = T_xM^n$ and $A = T_xM^n$, the tangent space is called *affine tangent space* and denoted by A_xM^n. The points "a" of A_xM^n are the tangent vectors at x.